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INTERACTION BETWEEN COPLANAR ELLIPTIC CRACKS—I. NORMAL LOADING

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Abstract—A recently developed integral equation method is used to solve the problem of interaction between coplanar, elliptic cracks under normal loading. The pair of dual integral equations, in a Cartesian system is first transformed to four sets of infinite systems of Fredholm integral equations of the second kind in a cylindrical polar coordinate system. For cracks which are well separated, a perturbed solution of these integral equations can be obtained in terms of the separation parameter β. Analytical expressions for the stress intensity factor and the strain energy of deformation per crack, when subjected to a constant normal loading, have been given up to the order β^6. Results have been illustrated graphically.

1. INTRODUCTION

The paper deals with the interaction of two coplanar elliptic cracks with their axes parallel and embedded in an infinite elastic medium subjected to normal loading. Kostrov and Das (1984), investigated the stress distribution due to a single elliptic crack in an infinite medium under shear, to gain insight into the mechanism leading to after-shock phenomena of an earthquake. In reality, any material surface contains a system of cracks rather than an isolated one. Rupture zones occurring in the neighbourhood of a seismic fault can be well approximated by elliptic cracks (Kostrov and Das, 1984). In material structures, pre-existing cracks interact to form major cracks leading to fracture. Thus, the study of interacting elliptic cracks subjected to a given set of external loads and environmental conditions, becomes extremely important for the purposes of design and safe-life prediction of material structures. In the seismological context, such studies may thus lead to a better prediction of after-shock phenomena.

Analytical studies of the interaction problem, even for coplanar cracks have so far been limited to two-dimensional ones. An extensive list of articles dealing with analytical solutions for these type of problems can be found in Sneddon and Lowengrub (1969).

The exact treatment of a regular array of three-dimensional cracks is rather complicated. Coplanar, equal, penny-shaped cracks under normal or shear loadings have been studied by several authors (Collins, 1963; Fu and Keer, 1969). Analytical studies involving elliptic cracks have been restricted to a single elliptic crack in an infinite medium (Kassir and Sih, 1975). [For recent work, see Nishioka and Atluri (1980).] For an elliptic crack in a half-space with a free surface a finite distance away, analytical solutions have been given recently by Roy and Chatterjee (1992). The corresponding numerical approach, for a semi-elliptic crack normal to the free surface can be found in Shah and Kobayashi (1973). The main inhibiting factor for analytical studies involving elliptic cracks, perhaps, has been the choice of an ellipsoidal coordinate system used to satisfy the conditions on the crack. The ellipsoidal coordinate is unsuitable for solving the interaction problem for elliptic cracks.

Nishitani and Murakami (1974) used a numerical approach for solving this problem. In their method, boundary conditions on the coplanar elliptic crack faces are determined by body force densities with appropriate weights. These unknown weights are determined by matching with the boundary conditions at a finite number of reference points on the crack surfaces. A similar numerical approach was followed by Isida *et al.* (1985) for parallel elliptic cracks under tension. Numerical results for coplanar, semi-elliptical cracks under tension have been given by Murakami and Nishitani (1981), and for bending by Murakami and Nemat-Nasser (1982).

In the present study, we approach the title problem from a different angle. Although the starting dual integral equation is the same, we give analytical solutions for these equations, as opposed to the numerical ones given in Nishitani and Murakami (1972). By means of suitable transformations similar to the ones in Roy and Chatterjee (1992), we transform the pair of dual integral equations in the Cartesian coordinate system, to a quadruplet of infinite systems of integral equations in the cylindrical polar coordinate system. Use of the integral equation method (Roy and Chatterjee, 1992) then allows these infinite systems to be reduced to four infinite systems of Fredholm integral equations of the second kind. For symmetrically applied normal loadings, these equations combine to a pair of infinite systems of Fredholm integral equations of the second kind.

When the crack faces are subjected to a constant normal load, these equations have been solved by a perturbation technique for various values of the parameter β ($\ll 1$). Analytical expressions for the stress intensity factor and the strain energy of deformation per crack have been obtained up to the order β^8 (Chatterjee, 1990). Here, for the sake of brevity we have presented the expressions only up to the order β^6. In the limiting case, $b/a = 1$, the results obtained by Collins (1963) for penny-shaped cracks are recovered.

The effect of the second elliptic crack on the first, for different crack positions has been illustrated by plots of the stress-intensity factor. Numerical results obtained in the present study from the analytical expressions agree well with those of Nishitani and Murakami (1974), obtained by numerical methods. It can be seen that up to the order β^6, the agreement is within 5% for closely spaced cracks ($\beta = 0.4$) when terms up to O (β^8) are taken into account the agreement is within 1%. For both cases, for well separated cracks ($0.1 \leqslant \beta \leqslant 0.4$) the agreement is within 1%.

The analytical solutions obtained here, can be easily extended to more realistic types of loading of the cracks and can thus be used for testing the accuracy and interpreting the solutions by any numerical methods, such as the boundary element, finite element methods or numerical methods based on the body force concept.

2. FORMULATION OF THE PROBLEM

Let the coplanar elliptic cracks, with their axes parallel, be subjected to normal pressures $\sigma_{zz}^{(j)}(x, y)$, and occupy the regions S_j ($j = 1, 2$) in an isotropic, infinite elastic medium (see Fig. 1):

$$S_1: \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1 \,;\; z = 0,$$

$$S_2: \frac{(x-f)^2}{a^2} + \frac{(y-g)^2}{b^2} \leqslant 1 \,;\; z = 0. \tag{1}$$

The symmetric nature of forces acting on both crack faces allows us to consider the equivalent half-space problem on $z \geqslant 0$, with components of normal stresses $\tau_{zi}^{(j)}(x, y, 0)$ and displacements $U_i^{(j)}(x, y, 0)$ ($j = 1, 2$; $i = x, y, z$) on the crack satisfying the following boundary conditions on $z = 0^+$ (henceforward, the $+$ sign will be dropped):

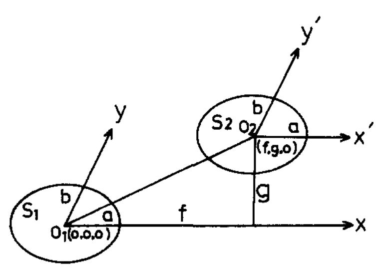

Fig. 1. The rectangular Cartesian coordinate system used to describe the geometry of a pair of equal, coplanar elliptic cracks, with axes parallel to each other.

For $j = 1, 2$

$$\tau_{zx}^{(j)}(x, y, 0) = \tau_{zy}^{(j)}(x, y, 0) = 0 \quad \forall\ (x, y), \tag{2a}$$

$$\tau_{zz}^{(j)}(x, y, 0) = -\sigma_{zz}^{(j)}(x, y) \quad \forall\ (x, y) \in S_j, \tag{2b}$$

$$u_z^{(j)}(x, y, 0) = 0 \quad \forall\ (x, y) \notin S_j. \tag{2c}$$

In (2), $\sigma_{zz}^{(j)}(x, y)$ are the initial stresses acting on the crack faces. Henceforward, the superscript j, unless stated otherwise will stand for $j = 1, 2$.

The displacement vector $\mathbf{u}$, for the problem under consideration, can be expressed in the region $z \geqslant 0$ in terms of a harmonic potential $\Phi(x, y, z)$ (Kassir and Sih, 1975) as:

$$\mathbf{u} = (1-2\nu)\nabla\Phi + z\nabla\left(\frac{\partial\Phi}{\partial z}\right) - (3-4\nu)\frac{\partial\Phi}{\partial z}\mathbf{e}_z, \tag{3}$$

where ν is the Poisson ratio for the medium and $\mathbf{e}_z$ is a unit vector in the $z > 0$ direction and $\Phi(x, y, z)$ satisfies

$$\nabla^2\Phi(x, y, z) = 0 \quad \text{on} \quad z \geqslant 0.$$

Seeking the solution of the above equation in the form:

$$\Phi(x, y, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \lambda^{-1} A(\xi, \eta) \exp[\mathrm{i}(\xi x + \eta y) - \lambda z]\, \mathrm{d}\xi\, \mathrm{d}\eta,$$

where $\lambda = (\xi^2 + \eta^2)^{1/2}$ and substituting in (3), one gets, after using the boundary condition (2c),

$$\frac{(1-\nu)}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\xi, \eta) \exp[\mathrm{i}(\xi x + \eta y)]\, \mathrm{d}\xi\, \mathrm{d}\eta = 0 \quad \forall\ (x, y) \notin S_j. \tag{4}$$

Let $w^{(j)}(x, y)$ denote normal displacement on the crack S_j. Then

$$\frac{(1-\nu)}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\xi, \eta) \exp[\mathrm{i}(\xi x + \eta y)]\, \mathrm{d}\xi\, \mathrm{d}\eta = w^{(j)}(x, y) \quad \forall\ (x, y) \in S_j. \tag{5}$$

Combining (4) and (5) and inverting, we get:

$$A(\xi, \eta) = \frac{1}{2\pi(1-\nu)}\left[\iint_{S_1} w^{(1)}(x', y') \exp[-i(\xi x' + \eta y')]\, \mathrm{d}x'\, \mathrm{d}y' \right.$$

$$\left. + \iint_{S_2} w^{(2)}(x'', y'') \exp[-\mathrm{i}(\xi x'' + \eta y'')]\, \mathrm{d}x''\, \mathrm{d}y''\right]. \tag{6}$$

Boundary conditions (2a) are automatically satisfied. The condition (2b) will be satisfied if $w^{(j)}(x, y)$ satisfy the following integral equation on $z = 0$:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\iint_{S_1} \lambda w^{(1)}(x', y') \exp \mathrm{i}[\xi(x-x') + \eta(y-y')]\, \mathrm{d}x'\, \mathrm{d}y'\, \mathrm{d}\xi\, \mathrm{d}\eta$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\iint_{S_2} \lambda w^{(2)}(x'', y'') \exp \mathrm{i}[\xi(x-x'') + \eta(y-y'')]\, \mathrm{d}x''\, \mathrm{d}y''\, \mathrm{d}\xi\, \mathrm{d}\eta$$

$$= \frac{2\pi(1-\nu)}{\mu}\begin{cases}\sigma_{zz}^{(1)}(x,y) & \forall\ (x,y)\in S_1, \\ \sigma_{zz}^{(2)}(x,y) & \forall\ (x,y)\in S_2,\end{cases} \quad \begin{matrix}(7a)\\(7b)\end{matrix}$$

where μ is the modulus of rigidity of the medium.

We note that the solutions $w^{(j)}(x, y)$ must satisfy appropriate edge conditions on S_j and the regularity conditions at infinity.

3. TRANSFORMATION OF THE INTEGRAL EQUATIONS

As a first step, we transform the integral equations (7) from the Cartesian to the cylindrical polar coordinate system by the following transformations:

$$(x, y, z) = (aX, bY, z). \tag{8}$$

The ellipses S_j then transform to circles S'_j:

$$S'_1: X^2+Y^2 \leqslant 1; \quad z=0,$$
$$S'_2: (X-f/a)^2+(Y-g/b)^2 \leqslant 1; \quad z=0. \tag{9}$$

The point (x, y) on the physical crack plane is represented as:

$$(x, y) = (ap\cos\theta, bp\sin\theta),$$
$$0<\rho<\infty, \quad 0\leqslant\theta\leqslant 2\pi. \tag{10}$$

A point on the crack face S'_j will then have the representation:

$$(X, Y) = \begin{cases}(r_1\cos\theta_1, r_1\sin\theta_1) & \text{on} \quad S_1, \\ (f/a+r_2\cos\theta_2, g/b+r_2\sin\theta_2) & \text{on} \quad S_2,\end{cases} \tag{11}$$

where (r_1, θ_1) and (r_2, θ_2) are polar coordinates with respect to the centres O_1, O_2 of the cracks S'_1 and S'_2 (see Fig. 2).

The following relations exist between r_1, r_2:

$$r_2^2 = r_1^2+d^2-2r_1d\cos(\theta_1-\alpha),$$
$$r_1^2 = r_2^2+d^2-2r_2d\cos(\pi-\theta_2+\alpha), \tag{12}$$

where

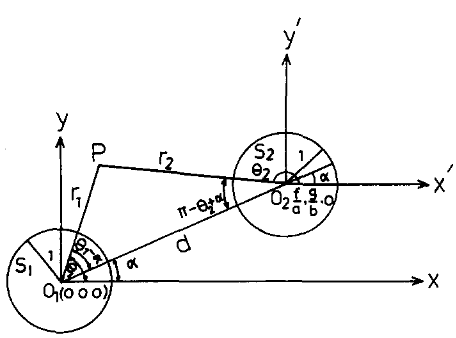

Fig. 2. Transformation of the rectangular Cartesian to the cylindrical polar coordinate system on a pair of equal, coplanar, elliptic cracks, with axes parallel to each other.

$$d = (f^2/a^2 + g^2/b^2)^{1/2},$$
$$\alpha = \tan^{-1}(ga/fb).$$

We make the further transformation:

$$(\xi a, \eta b) = (k\cos x, k\sin x),$$
$$0 < k < \infty; \quad 0 \leqslant x \leqslant 2\pi, \tag{13}$$

and make use of the following result (Gradshteyn and Ryzhik, 1980):

$$\exp(\pm i z\cos\theta) = \sum_{n=0}^{\infty} \varepsilon_n (\pm i)^n J_n(z)\cos n\theta,$$
$$\varepsilon_n = \begin{cases} 1, & n = 0, \\ 2, & n > 0. \end{cases} \tag{14}$$

Let us now assume that in the transformed coordinate system, the displacement and initial stress on the crack faces have the following Fourier series expansions:

$$[w^{(j)}(X,Y), \sigma_{zz}^{(j)}(X,Y)] = \sum_{n=0}^{\infty} [w_n^{(j)}(\rho), t_n^{(j)}(\rho)]\cos n\theta + \sum_{n=1}^{\infty} [\bar{w}_n^{(j)}(\rho), \bar{t}_n^{(j)}(\rho)]\sin n\theta. \tag{15}$$

Following the set of transformations indicated in (8), (10) and (13), then using (14), (15) and relations (A1) given in Appendix A, we separate out the Fourier cosine component terms on both sides of the resulting equation and obtain, for example, from (7a), the infinite system of integral equations on $z = 0$:

$\forall s = 0, 1, \ldots, \infty$; $(n+s)$ even; for each s, $(n+p)$ even; $r \in [0, 1]$:

$$\varepsilon_s \sum_{n=0}^{\infty} I_{n,s}^{C} \mathscr{L}_{n,s}[w_n^{(1)}(\rho)] + \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} (-1)^n \varepsilon_p e_s I_{n,p}^{C} \mathscr{L}_{n,p}[w_n^{(2)}(\rho) f(\alpha,k,d)]$$
$$+ \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} (-1)^n \varepsilon_p e_s I_{n,p}^{S} \mathscr{L}_{n,p}[\bar{w}_n^{(2)}(\rho)\bar{f}(\alpha,k,d)] = \frac{\pi(1-\nu)b}{\mu} t_s^{(1)}(r), \tag{16}$$

where ε_s has been defined in (14) and

$$e_s = \begin{cases} \frac{1}{2}, & s = 0, \\ 1, & s > 0, \end{cases}$$

$$f(\alpha,k,d) = \cos(p+s)\alpha J_{p+s}(kd) + (-1)^s \cos(p-s)\alpha J_{p-s}(kd),$$
$$\bar{f}(\alpha,k,d) = \sin(p+s)\alpha J_{p+s}(kd) + (-1)^s \sin(p-s)\alpha J_{p-s}(kd), \tag{17}$$

d and α are as defined in (12).

The other symbols used in (16) are:

$$\begin{pmatrix} I_{n,s}^{C} \\ I_{n,s}^{S} \end{pmatrix} = \tfrac{1}{2} i^s (-i)^n \int_0^{2\pi} (1 - k_0^2\cos^2 x)^{1/2} \begin{pmatrix} \cos nx \cos sx \\ \sin nx \sin sx \end{pmatrix} \mathrm{d}x, \tag{18}$$

$$k_0 = (1 - b^2/a^2)^{1/2},$$

$$\mathscr{L}_{n,s}[w_n^{(j)}(\rho)] = \int_0^\infty \int_0^1 pk^2 w_n^{(j)}(\rho) J_n(kp) J_s(kr_j)\,\mathrm{d}p\,\mathrm{d}k. \tag{19}$$

Note that $I_{n,s}^{C(S)} = 0$ whenever $(n+s)$ is odd. Hence the restriction on $(n+s)$ and $(n+p)$ imposed on (16). The system (16) will thus exist only when $(n+s)$ or $(n+p)$ is even.

Proceeding similarly with (7b), another such infinite system of integral equations results. Thus, on $z = 0$:

$s = 0, 1, \ldots, \infty$; $(n+s)$ even; for each s; $(n+p)$ even; $r \in [0, 1]$:

$$\varepsilon_s \sum_{n=0}^{\infty} (-1)^n I_{n,s}^C \mathscr{L}_{n,s}[w_n^{(2)}(\rho)] + \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \varepsilon_p e_s I_{n,p}^C \mathscr{L}_{n,p}[w_n^{(1)}(\rho) f(\alpha, k, d)]$$
$$+ \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \varepsilon_p e_s I_{n,p}^S \mathscr{L}_{n,p}[\bar{w}_n^{(1)}(\rho) \bar{f}(\alpha, k, d)] = \frac{2\pi(1-\nu)b}{\mu} (-1)^s t_s^{(2)}(r). \tag{20}$$

On equating the Fourier sine components likewise, two more infinite systems of integral equations result from (7a) and (7b). These have forms similar to (16) and (20) and can be written down from these by interchanging $w_n^{(j)}(\rho)$ and $\bar{w}_n^{(j)}(\rho)$; $I_{n,s}^C, I_{n,p}^C, f(\alpha, k, d)$ with $I_{n,s}^S, I_{n,p}^S, \bar{f}(\alpha, k, d)$ respectively and changing the "+" sign appearing before the second left term in (16) and (20) to the "−" sign. In these operations, one must simultaneously change the "+" sign before the second term in $f(\alpha, k, d), \bar{f}(\alpha, k, d)$ to the "−" sign.

Since all four infinite systems have identical structures, we shall proceed henceforward with (16) and (20) only.

4. FURTHER REDUCTION OF THE INFINITE SYSTEM OF INTEGRAL EQUATIONS

To reduce the infinite system of integral equations (16) and (20) and their corresponding Fourier sine component versions to tractable forms, we relate the Fourier components of the normal displacement on the crack faces with functions $\Phi_n^{(j)}(t)$, $\bar{\Phi}_n^{(j)}(t)$ through the relations:

$$[w_n^{(j)}(\rho), \bar{w}_n^{(j)}(\rho)] = \frac{2}{\pi} \rho^n \int_\rho^1 \frac{t^{-n}}{(t^2-\rho^2)^{1/2}} [\Phi_n^{(j)}(t), \bar{\Phi}_n^{(j)}(t)]\,\mathrm{d}t, \tag{21}$$

so that the inverse relation is:

$$[\Phi_n^{(j)}(t), \bar{\Phi}_n^{(j)}(t)] = -t^n \frac{\mathrm{d}}{\mathrm{d}t} \int_t^1 \frac{\rho^{1-n}}{(\rho^2-t^2)^{1/2}} [w_n^{(j)}(\rho), \bar{w}_n^{(j)}(\rho)]\,\mathrm{d}\rho, \tag{22}$$

The successive steps in the reduction are briefly described here, while details can be found in Roy and Chatterjee (1992). Separating out the sth integral equation from (16) for example, and multiplying both sides by r^{s+1} and integrating with respect to r within $[0, \xi]$, where $0 \leqslant \xi \leqslant 1$, one obtains:

$$\varepsilon_s I_{s,s}^C \int_0^\xi r^{s+1} \mathscr{L}_{s,s}[w_n^{(j)}(\rho)]\,\mathrm{d}r = \int_0^\xi r^{s+1} G(r)\,\mathrm{d}r,$$

where, $G(r)$ contains the interaction terms [that is the second and third left-hand terms in (16)], the remainder of the first left term (the terms in which $n \neq s$) and the right-hand term in (16).

Restoring various terms in the above equation by using (19) and utilizing standard integrals involving Bessel functions, listed as (A2) in Appendix A, we get:

$$\varepsilon_s I^C_{s,s} \int_0^\infty \int_0^1 pk\xi^{s+1} w_n^{(j)}(\rho) J_{s+1}(k\xi) J_s(kp)\,\mathrm{d}p\,\mathrm{d}k = \int_0^\xi r^{s+1} G(r)\,\mathrm{d}r.$$

Making use of (21) in the above equation, interchanging the order of integration between t and p and integrating over ρ with the help of another standard result on Bessel functions, listed as (A3), one obtains:

$$\left(\frac{2}{\pi}\right)^{1/2} \varepsilon_s I^C_{s,s} \int_0^\infty \int_0^1 (kt)^{1/2} \xi^{s+1} \Phi_s^{(j)}(t) J_{s+1}(k\xi) J_{s+1/2}(kt)\,\mathrm{d}t\,\mathrm{d}k = \int_0^\xi r^{s+1} G(r)\,\mathrm{d}r.$$

Evaluating the k-integral by using the result (A4), we finally get:

$$\frac{2}{\pi} \varepsilon_s I^C_{s,s} \int_0^\xi \frac{t^{s+1} \Phi_s^{(j)}(t)}{(\xi^2 - t^2)^{1/2}}\,\mathrm{d}t = \int_0^\xi r^{s+1} G(r)\,\mathrm{d}r,$$

which is an Abel type integral equation. Inversion leads to:

$$\varepsilon_s I^C_{s,s} \Phi_s^{(j)}(\xi) = \xi^{-s} \frac{d}{d\xi} \int_0^\xi \frac{t}{(\xi^2 - r^2)^{1/2}} \left[\int_0^t r^{s+1} G(r)\,\mathrm{d}r \right] \mathrm{d}t. \tag{23}$$

Carrying out necessary simplifications permitted by use of the results (A2)–(A4), the final form of the right-hand term in (23) is:

$$-\varepsilon_s \sum_{\substack{n=0\\ n\neq s}}^\infty I^C_{n,s} \int_0^1 L_{n,s}(\xi, t) \Phi_n^{(j)}(t)\,\mathrm{d}t - \sum_{n=0}^\infty \sum_{p=0}^\infty (-1)^n \varepsilon_p e_s I^C_{n,p} \int_0^1 K_{s,n,p}(\xi, t)$$

$$\Phi_n^{(i)}(t)\,\mathrm{d}t - \sum_{n=1}^\infty \sum_{p=1}^\infty (-1)^n \varepsilon_p e_s I^S_{n,p} \int_0^1 \bar{K}_{s,n,p}(\xi, t) \Phi_n^{(i)}(t)\,\mathrm{d}t$$

$$+ \frac{\pi(1-\nu)b}{\mu} \xi^{-s} \int_0^\xi \frac{r^{s+1}}{(\xi^2 - r^2)^{1/2}} t_s^{(j)}(r)\,\mathrm{d}r, \tag{24}$$

for $j = 1, i = 2$ and $j = 2, i = 1$. However, for $j = 2, \Phi_n^{(j)}(t)$ is to be replaced by $(-1)^n$ $\Phi_n^{(j)}(t)$ and $t_s^{(j)}(r)$ by $(-1)^s t_s^{(j)}(r)$.

The various symbols used in (24) are:

$$L_{n,s}(\xi, t) = (\xi t)^{1/2} \int_0^\infty k J_{n+1/2}(kt) J_{s+1/2}(k\xi)\,\mathrm{d}k,$$

$$K_{s,n,p}(\xi, t) = \cos(p+s)\alpha M^s_{n,p+s}(\xi, t) + (-1)^s \cos(p-s)\alpha M^s_{n,p-s}(\xi, t),$$

$$\bar{K}_{s,n,p}(\xi, t) = \sin(p+s)\alpha M^s_{n,p+s}(\xi, t) + (-1)^s \sin(p-s)\alpha M^s_{n,p-s}(\xi, t), \tag{25}$$

with

$$M^s_{n,p\pm s}(\xi, t) = (\xi t)^{1/2} \int_0^\infty k J_{s+1/2}(k\xi) J_{n+1/2}(kt) J_{p\pm s}(kd)\,\mathrm{d}k. \tag{26}$$

The sine component versions of the infinite system of Fredholm integral equations of the second kind to which (16) and (20) reduce are similar to (24). These may be written down from (24) by interchanging $I^C_{n,s}, I^C_{n,p}$ with $I^S_{n,s}, I^S_{n,p}$, respectively; $t_s^{(j)}(r)$, $\Phi_n^{(j)}(t)$ with $\bar{t}_s^{(j)}(r)$, $\bar{\Phi}_n^{(j)}(t)$ respectively and replacing $K_{s,n,p}(\xi, t)$, $\bar{K}_{s,n,p}(\xi, t)$ by $N_{s,n,p}(\xi, t)$, $\bar{N}_{s,n,p}(\xi, t)$, respectively, where

$$N_{s,n,p}(\xi,t) = \cos(p+s)\alpha M^{s}_{n,p+s}(\xi,t) - (-1)^{s}\cos(p-s)\alpha M^{s}_{n,p-s}(\xi,t),$$

$$\bar{N}_{s,n,p}(\xi,t) = \sin(p+s)\alpha M^{s}_{n,p+s}(\xi,t) - (-1)^{s}\sin(p-s)\alpha M^{s}_{n,p-s}(\xi,t). \qquad (27)$$

Also, the sign before the second term in (24) should be changed.

We now consider two particular ways of loading the crack faces.

Case A

The Fourier components of the prescribed initial stresses on the cracks are so related:

$$[t_s^{(1)}(r), \bar{t}_s^{(1)}(r)] = (-1)^s[t_s^{(2)}(r), \bar{t}_s^{(2)}(r)] = [t_s^{+}(r), \bar{t}_s^{+}(r)](say). \qquad (28)$$

Case B

Negative mirror symmetry: In this manner of loading the cracks, the Fourier components of prescribed normal stress are related by:

$$[t_s^{(1)}(r), \bar{t}_s^{(1)}(r)] = (-1)^{s+1}[t_s^{(2)}(r), \bar{t}_s^{(2)}(r)] = [t_s^{-}(r), \bar{t}_s^{-}(r)](say). \qquad (29)$$

It may be verified that:

$$[\Phi_n^{(1)}(t), \bar{\Phi}_n^{(1)}(t)] = (-1)^n[\Phi_n^{(2)}(t), \bar{\Phi}_n^{(2)}(t)] = [\Phi_n^{+}(t), \bar{\Phi}_n^{+}(t)](say) \qquad (30)$$

for Case A, whereas,

$$[\Phi_n^{(1)}(t), \bar{\Phi}_n^{(1)}(t) = (-1)^{n+1}[\Phi_n^{(2)}(t), \bar{\Phi}_n^{(2)}(t)] = [\Phi_n^{-}(t), \bar{\Phi}_n^{-}(t)](say) \qquad (31)$$

for Case B.

For both cases, however, the quadruplet of infinite system integral equations similar to (23) combine to just a pair of infinite systems of Fredholm integral equations of the second kind, involving $\Phi_n^{\pm}(t)$, $\bar{\Phi}_n^{\pm}(t)$ corresponding to which, the right-hand term will contain $t_s^{\pm}(r)$ or $\bar{t}_s^{\pm}(r)$, respectively. Specifically, these are:

$\forall\ s = 0, 1, \ldots, \infty$, $(n+s)$ even; for each s, $(n+p)$ even; $\xi, r \in [0, 1]$:

$$\varepsilon_s I^C_{s,s}\Phi_s^{\pm}(\xi) + \varepsilon_s \sum_{\substack{n=0\\ n\neq s}}^{\infty} I^C_{n,s}\int_0^1 L_{n,s}(\xi,t)\Phi_n^{\pm}(t)\,dt \pm e_s\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\varepsilon_p I^C_{n,p}\int_0^1 K_{s,n,p}(\xi,t)\Phi_n^{\pm}(t)\,dt \pm e_s\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\varepsilon_p I^S_{n,p}\int_0^1 \bar{K}_{s,n,p}(\xi,t)\bar{\Phi}_n^{\pm}(t)\,dt = F_s^{\pm}(\xi) \qquad (32)$$

and

$\forall\ s = 1, 2, \ldots, \infty$, $(n+s)$ even; for each s, $(n+p)$ even; $\xi, r \in [0, 1]$:

$$\varepsilon_s I^S_{s,s}\Phi^\pm_s(\xi)+\varepsilon_s\sum_{\substack{n=1\\n\neq s}}^{\infty} I^S_{n,s}\int_0^1 L_{n,s}(\xi,t)\Phi^\pm_n(t)\,\mathrm{d}t$$
$$\mp e_s\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\varepsilon_p I^S_{n,p}\int_0^1 \bar{N}_{s,n,p}(\xi,t)\Phi^\pm_n(t)\,\mathrm{d}t$$
$$\pm e_s\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\varepsilon_p I^C_{n,p}\int_0^1 N_{s,n,p}(\xi,t)\Phi^\pm_n(t)\,\mathrm{d}t=\bar{F}^\pm_s(\xi). \quad (33)$$

The symbols used in (31) and (32) have all been defined earlier, except:

$$[F^\pm_s(\xi),\bar{F}^\pm_s(\xi)]=\frac{\pi(1-v)b}{\mu}\xi^{-s}\int_0^\xi\frac{r^{s+1}}{(\xi^2-r^2)^{1/2}}[t^\pm_s(r),\ \bar{t}^\pm_s(r)]\,\mathrm{d}x. \quad (34)$$

Two alternate forms of the term $\int_0^1 L_{n,s}(\xi,t)g(t)\,\mathrm{d}t$ are being given below for later use:

(1) $n<s$, $(n+s)$ even:

On integration of the inner integral over t by parts and division of the range of integration $[0,1]$ into $[0,\xi]$, $[\xi,1]$ one finds on using the well-known result (A5), that the integral over $[\xi,1]$ vanishes since $n<s$. On repeating this procedure of integration $(s-n)/2$ times in succession, one obtains, finally, after making use of (A4):

$$\int_0^1 L_{n,s}(\xi,t)g(t)\,\mathrm{d}t=\frac{(-1)^{(s-n)/2}\xi^{-s}}{2^{(s-n)/2}\Gamma\left(\dfrac{s-n}{2}\right)}\int_0^\xi[D^{((s-n)/2)}(t^{-n-1}g(t))](\xi^2-t^2)^{\frac{s-n}{2}-1}t^{n+s+2}\,\mathrm{d}t, \quad (35)$$

where the operator $D\equiv(1/t)(\partial/\partial t)$.

(2) $n>s$, $(n+s)$ even:

Integrating $\int_0^1 L_{n,s}(\xi,t)g(t)\,\mathrm{d}t$, over t once by parts, we find by using (A5) that

$$\int_0^1 L_{n,s}(\xi,t)g(t)\,\mathrm{d}t=\xi g(\xi)G(\xi,\xi+0)+\xi^{1/2}\int_\xi^1 t^{-n-1}g(t)\frac{d}{dt}[t^{n+(3/2)}G(\xi,t)]\,\mathrm{d}t, \quad (36)$$

where

$$G(\xi,\xi+0)=\lim_{t\to\xi+0}G(\xi,t),$$

$$G(\xi,t)=\int_0^\infty J_{s+(1/2)}(k\xi)J_{n+(3/2)}(kt)\,\mathrm{d}k$$
$$=\frac{\xi^{s+(1/2)}\Gamma\left(\dfrac{n+s+3}{2}\right)}{t^{s+(3/2)}\Gamma\left(\dfrac{n-s+2}{2}\right)\Gamma(s+\frac{3}{2})}F\left(\frac{n+s+3}{2},\frac{s-n}{2};\,s+\tfrac{3}{2};\,\frac{\xi^2}{t^2}\right). \quad (37)$$

$F(\alpha,\beta,y;z)$ is the hypergeometric function of the first kind and the above form of $G(\xi,t)$ has been obtained by using (A5).

5. ANALYTICAL SOLUTION FOR TWO COPLANAR ELLIPTIC CRACKS UNDER TENSION

Although the infinite systems of Fredholm integral equations of the second kind (31) and (32) look formidable, a solution for a particular prescribed loading can be obtained when the cracks are well separated ($\beta \equiv d^{-1} \ll 1$). This solution is seen to be a perturbation on the corresponding single crack solution.

We note first that the infinite integrals in (32) and (33) can be expressed in closed form (Gradshteyn and Ryzhik, 1980) as follows:

$$M^s_{n,q}(\xi, t) = \frac{2\xi^{s+1}t^{n+1}\beta^{n+s+3}}{\Gamma(n+\frac{3}{2})\Gamma(s+\frac{3}{2})\Gamma\left(1-\frac{\delta}{2}\right)} F_4\left(\frac{\delta}{2}, \frac{\gamma}{2}; n+\tfrac{3}{2}, s+\tfrac{3}{2}; \beta^2 t^2, \beta^2\xi^2\right), \tag{38}$$

where $\beta = 1/d$ and for $q = p+s$, $\gamma = 2s+n+p+3$, $\delta = n-p+3$, $q = p-s$, $\gamma = n+p+3$, $\delta = 2s+n-p+3$, F_4 is the hypergeometric series of the fourth kind.

When $d \to \infty$ (the condition corresponding to the absence of the second crack), (32) and (33) give results for a single, normally loaded elliptic crack in an isotropic half-space.

For a prescribed stress which is a polynomial of the order L given by:

$$\sigma_{zz}(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} x^i y^j, \quad i+j \leqslant L$$

we have for the single crack,

$$t_s(r) = \bar{t}_s(r) = 0 \quad \forall s > L.$$

It can be verified that for such a case, $\Phi_n(t)$ will be a polynomial in ξ of degree $(L+1)$. Further, since the equations (32) and (33) are valid in [0, 1], the forms (35) and (36) of the second left terms in these equations suggest that to avoid the possible singularity at $t = 0$, $\Phi_n^{\pm}(\xi)$, $\bar{\Phi}_n^{\pm}(\xi)$ must have the following typical forms:

$$\Phi_n^{\pm}(\xi) = \begin{cases} \xi^{n+1}[a_0^n + a_2^n\xi^2 + a_4^n\xi^4 + \cdots + a_m^n\xi^m], & \forall\, n \leqslant L, \\ 0, & \forall\, n > L, \end{cases} \tag{39}$$

where

$$m = \begin{matrix} L-n, & \text{if } L \text{ is even}, \\ L+1-n, & \text{if } L \text{ is odd}. \end{matrix}$$

The relation (39) implies that the number of equations in the infinite systems (32) and (33) now become finite. The coefficients a_n^m can be obtained on equating like powered terms in ξ on both sides.

The form (39) of $\Phi_n^{\pm}(\xi)$ also follows from the fact that higher order coefficients a_m^n for $m > L$ if retained, will be related to each other with the terms on the right identically zero (in view of $t_s^{\pm}(r), \bar{t}_s^{\pm}(r) \equiv 0 \quad \forall\, s > L$) and will hence be identically zero. This result is in conformity with the well-known Galin's theorem (Kassir and Sih, 1975).

Based on the above remarks, we are now in a position to solve the crack interaction problem for prescribed polynomial loadings. We neglect initially the interaction terms in (32) and (33) and solve for $\Phi_n^{\pm}(\xi)$, $\bar{\Phi}_n^{\pm}(\xi)$ for the particular prescribed loading. Substitution of these values of $\Phi_n^{\pm}(\xi)$, $\bar{\Phi}_n^{\pm}(\xi)$ in the interaction terms in (32) and (33) with the use of (38) and making the assumption that $d \gg 1$ (that is, $\beta \ll 1$), these terms give rise to another polynomial stress, the order of which will depend on the order of β^n retained. This polynomial stress, in addition to the initial prescribed stress is now assumed to be the stress acting on the first crack in the infinite medium. The new values of $\Phi_n^{\pm}(\xi)$, $\bar{\Phi}_n^{\pm}(\xi)$ can again be computed by equating like powered terms in ξ present in the assumed polynomial form of these functions as given in (39) and those occurring in the combined (interaction plus initial) stress terms on the right of (32) and (33). Proceeding in this manner, successive perturbation terms of the interaction problem can be obtained. Although we have obtained solutions up to the order β^8 (Chatterjee, 1990), here we present results for terms up to the order β^6, merely for the sake of brevity.

As an example, we consider the case of elliptic cracks under tension, specified by

$$\sigma_{zz}^{(1)}(x, y) = \sigma_{zz}^{(2)}(x, y) = p_0. \tag{40}$$

When this loading is applied to the cracks in the manner stated in Case A [see eqn (30)], then, dropping the + sign for convenience, the obtained solutions up to the order β^6 are:

$$\begin{aligned}
\Phi_0(\xi) &= \gamma_0\xi[A_0^0 + A_2^0\xi^2] + O(\beta^7),\\
\Phi_1(\xi) &= \frac{2\gamma_0 I_{0,0}^C}{3\pi I_{1,1}^C}\beta^4\xi^2[A_1^1 + A_3^1\xi^2] + O(\beta^7),\\
\bar{\Phi}_1(\xi) &= \frac{2\gamma_0 I_{0,0}^C}{3\pi I_{1,1}^S}\beta^4\xi^2[\bar{A}_1^1 + \bar{A}_3^1\xi^2] + O(\beta^7),\\
\Phi_2(\xi) &= \frac{2\gamma_0 I_{0,0}^C}{15\pi\Omega_0}\beta^5\xi^3 A_2^2 + O(\beta^7),\\
\bar{\Phi}_2(\xi) &= \frac{2\gamma_0 I_{0,0}^C}{15\pi I_{2,2}^S}\beta^5\xi^3\bar{A}_2^2 + O(\beta^7),\\
\Phi_3(\xi) &= \frac{2\gamma_0 I_{0,0}^C}{21\pi\Omega_1}\beta^6\xi^4 A_3^3 + O(\beta^7),\\
\bar{\Phi}_3(\xi) &= \frac{2\gamma_0 I_{0,0}^C}{21\pi\bar{\Omega}_1}\beta^6\xi^4\bar{A}_3^3 + O(\beta^7);\ \Phi_s(\xi), \bar{\Phi}_s(\xi) \sim O(\beta^7) \quad \forall\, s > 3.
\end{aligned} \tag{41}$$

In the above expressions:

$$\gamma_0 = \pi(1-\nu)bp_0/[\mu I_{0,0}^C].$$

The remaining symbols used have been explained in Appendix B.

6. NUMERICAL RESULTS

The quantities of physical interest in crack interaction problems are the stress intensity factors on the edge of the cracks and the strain energy of deformation per crack resulting from the prescribed loadings applied on the crack faces.

For normally loaded elliptic cracks, there is only one stress intensity factor $k_1(\phi)$ at the point ($a\cos\phi$, $b\sin\phi$) on the crack border. This is defined by (Roy and Chatterjee, 1992):

$$k_1(\phi) = \frac{\mu}{\pi(1-\nu)b}\left(\frac{b}{a}\right)^{1/2}(a^2\sin^2\phi + b^2\cos^2\phi)^{1/4}\left[\sum_{n=0}^{\infty}\Phi_n(1)\cos n\phi + \sum_{n=1}^{\infty}\bar{\Phi}_n(1)\sin n\phi\right]. \tag{42}$$

It may be noted that the dependence of $k_1(\phi)$ on the elastic properties μ (rigidity) and ν (Poisson's ratio) vanishes because of the cancellation of the factor $\mu(1-\nu)^{-1}$, when $\Phi_n(1)$, $\bar{\Phi}_n(1)$ obtained as in (41) are substituted into the above expression.

In order to highlight the effect that the second elliptic crack a finite distance away from the first has on the stress intensity factor, the stress magnification factor $M(\phi)$ is defined as the ratio:

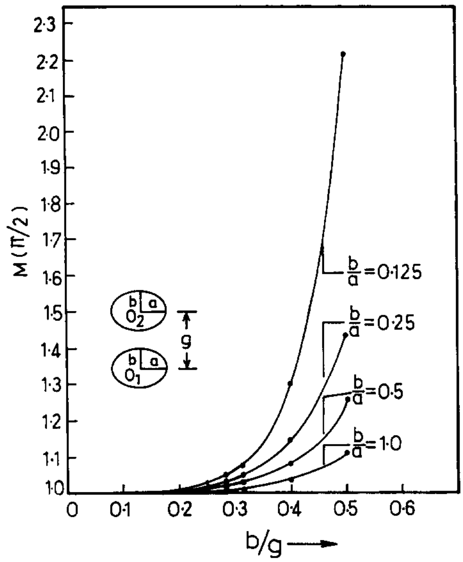

Fig. 3. Variation of the stress magnification factor $M(\phi)$ at the point $B(\phi = \pi/2)$ with $\beta = b/g$ for various pairs of ellipses with aspect ratios 1.0, 0.5, 0.25, 0.125, arranged along their minor axes.

$$M(\phi) = k_1(\phi)/k_*(\phi), \tag{43}$$

where

$$k_*(\phi) = \frac{p_0 b^{1/2}}{E(k_0)}(1-k_0^2\cos^2\phi)^{1/4}, \tag{44}$$

can be recognized as the stress intensity factor at the point ϕ of a single normally loaded elliptic crack in an infinite medium.

In Fig. 3, the variation of $M(\phi)$ at the point $B(\phi = \pi/2)$ with the interaction parameter $\beta = b/g$ has been shown for pairs of ellipses having aspect ratios 0.5, 0.25 and 0.125 arranged (as shown in the inset) along their minor axis. The curve for penny-shaped cracks ($b/a = 1$) has been included for comparison. It can be concluded from the figure, that the stress magnification increases as the ellipses come closer and also as they become narrower.

In Table 1, values of $M(\phi)$ at the point B have been tabulated for various values of $\beta = b/g$ for the various crack pairs. Also given in this table are values of $M(B)$ computed by Nishitani and Murakami (1974) using a different approach. The two values agree within 5% in most cases.

It should be noted that the ellipses contact at the point B when $g = 2b$, that is for $\beta = 0.5$. Since under this condition, the exponent of the stress singularity changes sign, the stress intensity factor becomes physically meaningless. Consequently, we have allowed the cracks to approach each other but never contact, which means that β must always be less

Table 1. Stress magnification for pairs of ellipses arranged along their minor axis

	b/a			
$\beta = b/g$	1.0	0.5	0.25	0.125
0.476	1.088	1.168	1.343	1.891
0.40	1.041	1.083	1.150	1.302
	1.068†	1.126†	1.180†	1.218†
0.3125	1.015	1.033	1.053	1.079
	1.108†	1.038†	1.063†	1.082†
0.25	1.006	1.015	1.023	1.031
	1.007†	1.016†	1.028	1.040†
0.10	1.0003	1.0007	1.001	1.002

†Values obtained by Nishitani and Murakami (1974).

Table 2. Effect of inclusion of higher order terms in computation of $M(\phi)$ at B for a pair of ellipses with aspect ratio 0.5. Percentage of error with respect to previous results (Nishitani and Murakami, 1974) given in brackets

	$M(\phi)$ at B			
$\beta = b/g$	Nishitani and Murakami (1974)	$O(\beta^6)$	$O(\beta^7)$	$O(\beta^8)$
0.476	–	1.168	1.226	1.254
0.4	1.126	1.083 (4.3%)	1.1 (2.6%)	1.108 (1.8%)
0.3125	1.038	1.032 (0.6%)	1.0355 (0.25%)	1.0365 (0.15%)
0.25	1.016	1.014 (0.2%)	1.0151 (0.09%)	1.0153 (0.07%)
0.1	—	1.0007	1.0007	1.0007

than 0.5. Thus, the maximum value of β we have chosen in order to illustrate our result is $\beta = 0.476$ corresponding to $g = 2.1b$.

In order to get an idea of the rate of convergence of the series solution involving (41), we have computed numerically, the contributions of the terms of order β^7 and β^8, using expressions given in Chatterjee (1990). Results of the effect of inclusion of these terms on the stress magnification have been presented in Table 2, highlighted for a pair of ellipses with aspect ratio 0.5 and relevant values of $\beta = b/g$. A look at these values shows that for $\beta < 0.3$, the neglect of the β^7 and β^8 order terms introduces errors of less than 0.5% and 0.25%, respectively. For $\beta = 0.4$, the neglect of β^7, β^8 order terms introduces errors of less than 4.5% and 3.0%, respectively. This error is less than 1% for $g > 3b$.

The close agreement of the analytical and numerical results validates the theory behind the present method and opens up the possibility of an alternative and direct way of studying the effects of crack interaction.

For pairs of cracks having aspect ratios 0.5, 0.25, 0.125, arranged along their major axis, the stress magnification factor $M(\phi)$ at the closest point A ($\phi = 0$) varies with the crack interaction parameter $\beta = a/f$ as shown in Fig. 4. The values for $b/a = 1$ have again been plotted alongside for comparison. As seen from the figure, for this arrangement of cracks, the stress magnification increases as the cracks come closer and also as the ellipses become broader. The corresponding values of $M(\phi)$ at A ($\phi = 0$) have been tabulated in Table 3.

A comparison of the values of $M(\phi)$ listed in Tables 1 and 3 shows that at the closest point between the cracks, the stress magnification for cracks arranged along their major axis is less than the corresponding value for cracks arranged along their minor axis.

The non-dimensionalised stress-intensity factor $k_1(\phi)/(p_0 b^{1/2})$ gives an idea of the variation along the crack border. In Fig. 5, the variation of $k_1(\phi)/(p_0 b^{1/2})$ with ϕ has been

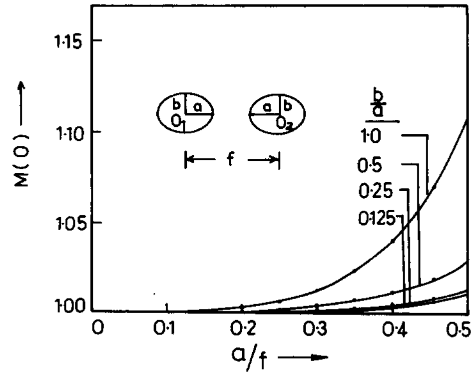

Fig. 4. Variation of the stress magnification factor $M(\phi)$ at the point $A(\phi = 0°)$ with $\beta = a/f$ for various pairs of ellipses with aspect ratios 1.0, 0.5, 0.25, 0.125, arranged along their major axes.

Table 3. Stress magnification at point A for pairs of ellipses arranged along their major axis

	b/a			
$\beta = a/f$	1.0	0.5	0.25	0.125
0.476	1.088	1.023	1.011	1.009
0.4	1.041	1.012	1.005	1.004
0.3125	1.015	1.005	1.002	1.001
0.25	1.006	1.002	1.001	1.0004
0.1	1.0003	1.0001	1.00003	1.00001

shown for a pair of ellipses with aspect ratio 0.5 and for various separation distances $\beta = b/g$ when arranged along their minor axis.

To illustrate the effect of the aspect ratio of the crack pairs at specified distances from each other, on the variation of $k_1(\phi)/(p_0 b^{1/2})$, Figs 6 and 7 have been provided for the two types of crack arrangements.

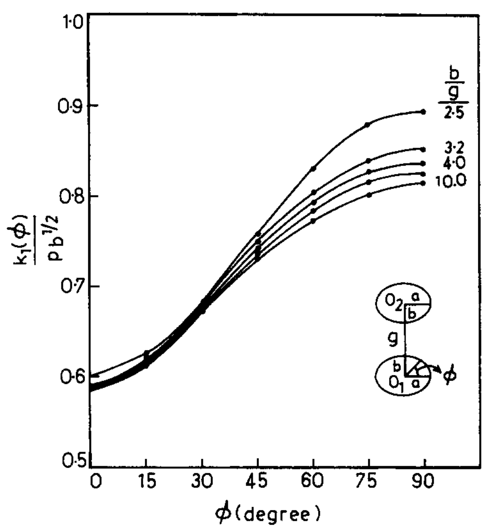

Fig. 5. Variation of the non-dimensionalized stress intensity factor $k_1(\phi)/p_0 b^{1/2}$ with ϕ for a pair of ellipses with aspect ratio $b/a = 0.5$ arranged along their minor axes at some fixed separation distances.

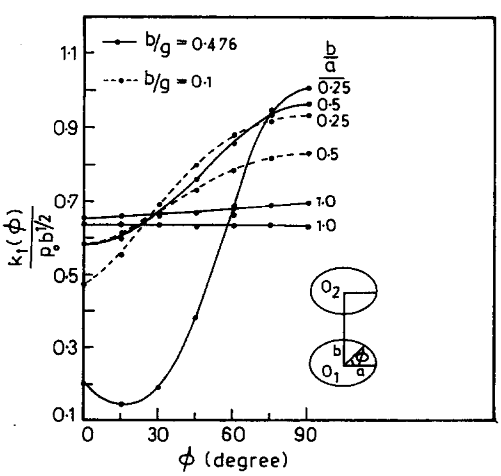

Fig. 6. Variation of $k_1(\phi)/p_0 b^{1/2}$ with ϕ for various pairs of ellipses ($b/a = 1.0, 0.5, 0.25$) at fixed distances along their minor axes.

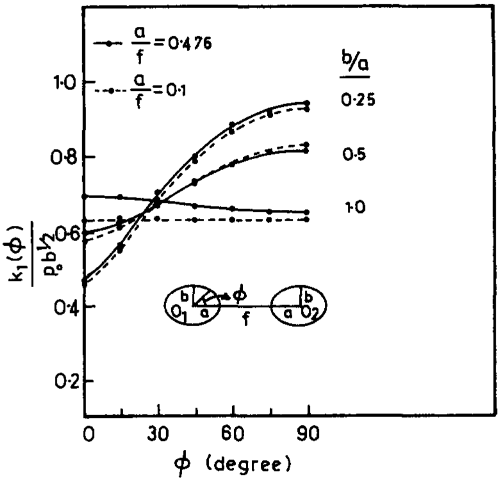

Fig. 7. Same as in Fig. 6, except ellipses are now arranged along their major axes.

The effect of staggering the cracks, for a fixed vertical spacing $g = 2b$, on the quantity $k_1(\phi)/(p_0 b^{1/2})$ along the crack border has been illustrated in Fig. 8. The figure reveals the asymmetric nature of the stress intensity factor for staggered cracks.

Finally, we mention that the increase in the strain energy of deformation per crack has been computed from:

$$S_z = \iint_{S_j} \sigma_{zz}^{(j)}(x, y) u_z^{(j)}(x, y, 0)\, dS_j$$

$$= 4abp_0 \int_0^1 t\Phi_0(t)\, dt. \tag{45}$$

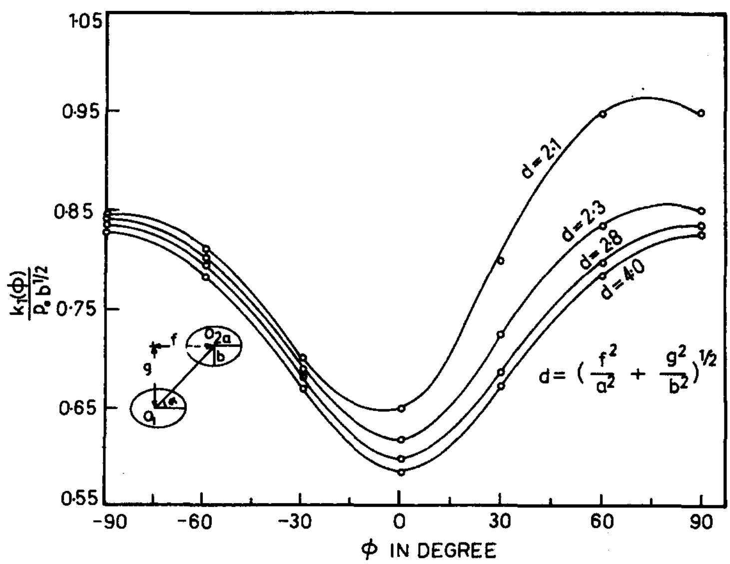

Fig. 8. The effect of staggering on $k_1(\phi)/p_0 b^{1/2}$ for pairs of ellipses with $b/a = 0.5$ arranged along their minor axes at a fixed distance $g/b = 2$.

Table 4. The increase in the strain energy of deformation per crack for various crack pairs. Values in brackets correspond to cracks arranged along their major axis

β	b/a			
	1.0	0.5	0.25	0.125
0.476	1.377	0.456 (0.4355)	0.011 (0.1225)	−0.024 (0.032)
0.4	1.357	0.446 (0.433)	0.124 (0.122)	0.010 (0.032)
0.3125	1.343	0.439 (0.433)	0.123 (0.122)	0.031 (0.032)
0.25	1.338	0.436 (0.432)	0.123 (0.122)	0.031 (0.032)
0.1	1.334	0.432 (0.432)	0.122 (0.122)	0.032 (0.032)
Single elliptic crack	1.333	0.432	0.122	0.032

The non-dimensional quantity

$$S_z^* = \mu S_z/[(1-\nu)p_0^2 a^3]$$

has been computed for various crack pairs and the results are tabulated in Table 4.

7. CONCLUSION

Analytical expressions for the stress intensity factor for coplanar elliptic cracks give as accurate values as those obtained by numerical solutions based on the body force method. The analytical solution can easily be adopted to obtain results for various positions of the second crack with respect to the first and for various aspect ratios without elaborate computations, while in numerical solutions, each individual case has to be considered separately. Analytical methods can also be easily extended to the case of multiple cracks. Although we have given results only for the constant normal loading, for other general types of loads, analytical results can be obtained without complication. Thus, for instance, for shear loading, a similar approach may be adopted to obtain an analytical perturbed solution. This has already been completed and will be reported in a later publication. Direct, numerical solution following Nishitani and Murakami (1974) and others will be quite formidable for this latter class of problem on account of the governing dual integral equations occurring as coupled pairs.

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APPENDIX A

Standard results on Bessel functions used in the text:

(1) (Gradshteyn and Ryzhik, 1980):

$$\exp(i\nu\psi)J_\nu(mR) = \sum_{k=-\infty}^{\infty} J_k(m\rho)J_{\nu+k}(mr)\exp(ik\phi), \tag{A1}$$

where $r > 0, p > 0, \phi > 0, R = (r^2+p^2-2rp\cos\phi)^{1/2}$ and ψ is the angle opposite to the side p, ν is an integer, m an arbitrary complex number.

(2) (Gradshteyn and Ryzhik, 1980):

$$\left(\frac{d}{Z\,dZ}\right)^m (Z^\nu J_\nu(Z)) = Z^{\nu-\mu}J_{\nu-\mu}(Z),$$

whence

$$\frac{d}{dr}[r^{n+1}J_{n+1}(kr)] = kr^{n+1}J_n(kr). \tag{A2}$$

(3) (Gradshteyn and Ryzhik, 1980):

$$\int_0^1 x^{\nu+1}(1-x^2)^\mu J_\nu(bx) = 2^\mu\Gamma(\mu+1)b^{-(\mu+1)}J_{\nu+\mu+1}(b), \quad b > 0, \quad \mathrm{Re}\,\nu > -1, \quad \mathrm{Re}\,\mu > -1. \tag{A3}$$

(4) (Gradshteyn and Ryzhik, 1980):

$$\int_0^\infty J_{\nu+1}(\alpha t)J_\mu(\beta t)t^{\mu-\nu}\,dt = \begin{cases} 0, & \alpha < \beta, \\ \dfrac{(\alpha^2-\beta^2)^{\nu-\mu}\beta^\mu}{2^{\nu-\mu}\alpha^{\nu+1}(\nu-\mu+1)}, & a \geqslant \beta, \end{cases} \quad \mathrm{Re}\,\mu, \quad \mathrm{Re}\,(\nu+1) > 0. \tag{A4}$$

(5) (Gradshteyn and Ryzhik, 1980):

For $0 < \beta < \alpha$,

$$\int_0^\infty J_\nu(\alpha t)J_\mu(\beta t)t^{-\lambda}\,dt$$

$$= \begin{cases} 0, \quad \mathrm{Re}\,(\nu-\mu+\lambda+1) < 0, \\ \dfrac{\beta^\mu\Gamma\left(\dfrac{\nu+\mu-\lambda+1}{2}\right)}{2^\lambda\alpha^{\mu-\lambda+1}\Gamma\left(\dfrac{\nu-\mu+\lambda+1}{2}\right)\Gamma(\mu+1)} F\left(\dfrac{\nu+\mu-\lambda+1}{2}, \dfrac{-\nu+\mu-\lambda+1}{2}; \mu+1; \dfrac{\beta^2}{\alpha^2}\right), \quad \mathrm{Re}\,(\nu-\mu+\lambda+1) > 0. \end{cases} \tag{A5}$$

APPENDIX B

List of symbols used in (41):

$$A_0^0 = 1+\frac{2q_0}{3\pi}\beta^3+\left(3q_1-\frac{I_{0,2}^C\alpha_0}{\Omega_0}\right)\frac{\beta^5}{5\pi}+\frac{4q_0^2}{9\pi^2}\beta^6+O(\beta^7),$$

$$A_2^0 = \frac{I_{0,0}^C\beta_0}{3\pi\Omega_0}\beta^5+O(\beta^7),$$

$$A_1^1 = r_0+\frac{3}{2}\left(r_1-\frac{5I_{1,3}^C\alpha_1}{\Omega_1}\right)\beta^2+O(\beta^4),$$

$$A_3^1 = \frac{1}{2}\frac{I_{1,1}^C}{\Omega_1}\beta_1\beta^2+O(\beta^4),$$

$$A_2^2 = \alpha_0, \quad A_3^3 = \alpha_1,$$

$$\bar{A}_1^1 = \bar{r}_0 + \frac{3}{2}\left(\bar{r}_1 - \frac{5I_{1,3}^S\bar{\alpha}_1}{\bar{\Omega}_1}\right)\beta^2 + O(\beta^4),$$

$$\bar{A}_3^1 = \frac{1}{2}\frac{I_{1,1}^S}{\bar{\Omega}_1}\bar{\beta}_1\beta^2 + O(\beta^4),$$

$$\bar{A}_2^2 = 5\bar{f}_0 + O(\beta^2), \quad \bar{A}_3^3 = \bar{\alpha}_1,$$

$$\alpha_0 = 3q_1 I_{0,2}^C + 5f_0 I_{0,0}^C, \quad \beta_0 = 3q_1 I_{2,2}^C + 5f_0 I_{0,2}^C,$$

$$\alpha_1 = 3\bar{r}_1 I_{1,3}^C + 7s_0 I_{1,1}^C, \quad \beta_1 = 3r_1 I_{3,3}^C + 7s_0 I_{1,s}^C,$$

$$\bar{\alpha}_1 = 3r_1 I_{1,3}^S + 7\bar{s}_0 I_{1,1}^S, \quad \bar{\beta}_1 = 3\bar{r}_1 I_{3,3}^S + 7\bar{s}_0 I_{1,3}^S,$$

$$\Omega_0 = I_{0,0}^C I_{2,2}^C - (I_{0,2}^C)^2,$$

$$\Omega_1 = I_{1,1}^C I_{3,3}^C - (I_{1,3}^C)^2,$$

$$\bar{\Omega}_1 = I_{1,1}^S I_{3,3}^S - (I_{1,3}^S)^2,$$

$$f_0 = \cos 2\alpha + (1/15)\sum_{m=1}^{\infty}(1-4m^2)I_{0,2m}[(3+2m)(5+2m)\cos\{2(m+1)\alpha\} + (3-2m)(5-2m)\cos\{2(m-1)\alpha\}],$$

$$q_0 = 1 + 2\sum_{m=1}^{\infty}(1-4m^2)I_{0,2m}\cos 2m\alpha,$$

$$q_1 = 1 + (2/9)\sum_{m=1}^{\infty}(1-4m^2)(9-4m^2)I_{0,2m}\cos 2m\alpha,$$

$$r_0 = \cos\alpha + (1/3)\sum_{m=1}^{\infty}(1-4m^2)I_{0,2m}[(3+2m)\cos\{(2m+1)\alpha\} + (3-2m)\cos\{(2m-1)\alpha\}],$$

$$r_1 = \cos\alpha + (1/45)\sum_{m=1}^{\infty}(1-4m^2)(9-4m^2)I_{0,2m}[(5+2m)\cos\{(2m+1)\alpha\} + (5-2m)\cos\{(2m-1)\alpha\}],$$

$$s_0 = \cos 3\alpha + (1/105)\sum_{m=1}^{\infty}(1-4m^2)I_{0,2m}[(3+2m)(5+2m)(7+2m)\cos\{(2m+3)\alpha\} + (3-2m)(5-2m)(7-2m)\cos\{(2m-3)\alpha\}],$$

where

$$I_{0,2m} = I_{0,2m}^C / I_{0,0}^C.$$

The expressions for $\bar{f}_0$, $\bar{r}_0$, $\bar{r}_1$, $\bar{s}_0$ can be written down from the corresponding expressions listed above on replacing, $\cos(p\alpha)$ by $\sin(p\alpha)$, $\cos\{(p+s)\alpha\}$ by $\sin\{(p+s)\alpha\}$ and $\cos\{(p-s)\alpha\}$ by $-\sin\{(p-s)\alpha\}$.